

Tight lower bound for the channel assignment problem

Arkadiusz Socała *

University of Warsaw, Poland

Abstract

We study the complexity of the CHANNEL ASSIGNMENT problem. A major open problem asks whether CHANNEL ASSIGNMENT admits an $O(c^n)$ -time algorithm, for a constant c independent of the weights on the edges. We answer this question in the negative i.e. we show that there is no $2^{o(n \log n)}$ -time algorithm solving CHANNEL ASSIGNMENT unless the Exponential Time Hypothesis fails. Note that the currently best known algorithm works in time $O^*(n!) = 2^{O(n \log n)}$ so our lower bound is tight.

*E-mail: as277575@students.mimuw.edu.pl

1 Introduction

In the CHANNEL ASSIGNMENT problem, we are given a symmetric weight function $w : V^2 \rightarrow \mathbb{N}$ (we assume that $0 \in \mathbb{N}$). The elements of V will be called vertices (as w induces a graph on the vertex set V with edges corresponding to positive values of w). We say that w is ℓ -bounded when for every $x, y \in V$ we have $w(x, y) \leq \ell$. An assignment $c : V \rightarrow \mathbb{Z}$ is called *proper* when for each pair of vertices x, y we have $|c(x) - c(y)| \geq w(x, y)$. The number $(\max_{v \in V} c(v) - \min_{v \in V} c(v) + 1)$ is called the *span* of c . The goal is to find a proper assignment of minimum span. Note that the special case when w is 1-bounded corresponds to the classical graph coloring problem. It is therefore natural to associate the instance of the channel assignment problem with an edge-weighted graph $G = (V, E)$ where $E = \{uv : w(u, v) > 0\}$ with edge weights $w_E : E \rightarrow \mathbb{N}$ such that $w_E(xy) = w(x, y)$ for every $xy \in E$ (in what follows we abuse the notation slightly and use the same letter w for both the function defined on V^2 and E). The minimum span is called also the span of (G, w) and denoted by $\text{span}(G, w)$.

It is interesting to realize the place CHANNEL ASSIGNMENT in a kind of hierarchy of constraint satisfaction problems. We have already seen that it is a generalization of the classical graph coloring. It is also a special case of the constraint satisfaction problem (CSP). In CSP, we are given a vertex set V , a constraint set \mathcal{C} and a number of colors d . Each constraint is a set of pairs of the form (v, t) where $v \in V$ and $t \in \{1, \dots, d\}$. An assignment $c : V \rightarrow \{1, \dots, d\}$ is *proper* if every constraint $A \in \mathcal{C}$ is satisfied, i.e. there exists $(v, t) \in A$ such that $c(v) \neq t$. The goal is to determine whether there is a proper assignment. Note that CHANNEL ASSIGNMENT corresponds to CSP where $d = s$ and every edge uv of weight $w(uv)$ in the instance of CHANNEL ASSIGNMENT corresponds to the set of constraints of the form $\{(u, t_1), (v, t_2)\}$ where $|t_1 - t_2| < w(uv)$.

In the general case the best known algorithm runs in $O^*(n!)$ time (see McDiarmid [7]). However, there has been some progress on the ℓ -bounded variant. McDiarmid [7] came up with an $O^*((2\ell + 1)^n)$ -time algorithm which has been next improved by Kral [6] to $O^*((\ell + 2)^n)$, further to $O^*((\ell + 1)^n)$ by Cygan and Kowalik [2] and to $O^*((2\sqrt{\ell + 1})^n)$ by Kowalik and Socala [5]. These are all dynamic programming (and hence exponential space) algorithms. The last but one applies the fast zeta transform to get a minor speed-up and the last one uses the meet-in-the-middle approach. Interestingly, all these works show also algorithms which *count* all proper assignments of span at most s within the same running time (up to polynomial factors) as the decision algorithm.

Since graph coloring is solvable in time $O^*(2^n)$ [1] it is natural to ask whether CHANNEL ASSIGNMENT is solvable in time $O^*(c^n)$, for some constant c . It is a major open problem (see [6, 2, 3]) to find such a $O(c^n)$ -time algorithm for c independent of ℓ or prove that it does not exist under a reasonable complexity assumption. A complexity assumption commonly used in such cases is the Exponential Time Hypothesis (ETH), introduced by Impagliazzo, Paturi and Zane [4]. It states that 3-CNF-SAT cannot be computed in time $2^{o(n)}$, where n is the number of variables in the input formula. The open problem mentioned above becomes even more interesting when we realize that under ETH, CSP does not have a $O^*(c^n)$ -time algorithm for a constant c independent of d , as proved by Traxler [8].

Our Results. Our main result is a proof that CHANNEL ASSIGNMENT does not admit a $O(c^n)$ -time for a constant c under the ETH assumption. By applying a sequence of reductions (see Figure 1) starting in 3-CNF-SAT and ending in CHANNEL ASSIGNMENT we were able to solve this open problem and to show that there is no $2^{o(n \log n)}$ -time algorithm solving CHANNEL ASSIGNMENT unless the ETH fails. Note that the currently best known algorithm works in time $O^*(n!) = 2^{O(n \log n)}$ so our lower bound is tight.

were two different variables. Note that in a 3-CNF-SAT instance with n variables and m clauses we have $3m$ occurrences of the n variables so there are 2^{3m} assignments of the occurrences.

We would like to represent two useful subsets of the set of all 2^{3m} assignments of the occurrences. The first is the set of the *consistent* assignments i.e. such assignments of the occurrences that all the occurrences of the same variable have the same value. The second is the set of the assignments of the occurrences such that every clause is satisfied (although they are allowed to have different values for different occurrences of the same variable i.e. they do not need to be consistent). Note that the instance of 3-CNF-SAT is a YES-instance if and only if the intersection of these two sets is nonempty.

To represent those two sets we would like to use the following concept. For a function $f : [a] \times [b] \rightarrow \mathbb{N}$ we define $X_f = \{\sum_{i=1}^a f(i, \sigma(i)) \mid \sigma : [a] \rightarrow [b]\}$. We call this set an *f-family*.

We will define a function f such that the elements of the f -family X_f correspond to the assignments of the occurrences such that every two occurrences of the same variable have the same value. Then we define another function g such that the g -family X_g represents the assignments of the occurrences such that every clause is satisfied. Thus we reduce 3-CNF-SAT into the following problem:

FAMILY INTERSECTION

Input: A function $f : [a] \times [b] \rightarrow \mathbb{N}$ and a function $g : [c] \times [d] \rightarrow \mathbb{N}$.

Question: Does $X_f \cap X_g \neq \emptyset$?

Example 1. Let us illustrate our approach on a 2-CNF-SAT formula $\varphi = (\alpha \vee \beta) \wedge (\neg \alpha \vee \gamma)$. We can make a distinction between different occurrences of the same variables $\varphi' = (\alpha_1 \vee \beta_1) \wedge (\neg \alpha_2 \vee \gamma_1)$. So we have four occurrences of the variables and $2^4 = 16$ assignments. We represent those assignments as numbers from the set $\{0, 1, \dots, 2^4 - 1\}$. However, it will be convenient to refer to these numbers as bit vectors of length 4 where the i -th bit represents the value of the i -th occurrence (among all the occurrences of all the variables).

To represent the set of the consistent assignments of the occurrences we can use a function $f : [n] \times [2] \rightarrow \mathbb{N}$ such that the value of $f(i, 1)$ is a bit vector representing all the occurrences of the i -th variable and $f(i, 2) = 0$. So in our example we have $f(1, 1) = 1010_2$, $f(2, 1) = 0100_2$, and $f(3, 1) = 0001_2$. Therefore $X_f = \{0000_2, 0001_2, 0100_2, 0101_2, 1010_2, 1011_2, 1110_2, 1111_2\}$.

To represent the set of the assignments which satisfies all the clauses we can use a function $g : [m] \times [3] \rightarrow \mathbb{N}$ such that $g(i, j)$ is a j -th assignment (in some fixed order) of the occurrences of the variables in the i -th clause which satisfies this clause. Note that every clause in 2-CNF-SAT have 3 assignments of the occurrences which satisfies this clause. So in our example we have $g(1, 1) = 1000_2$, $g(1, 2) = 0100_2$, $g(1, 3) = 1100_2$, $g(2, 1) = 0000_2$, $g(2, 2) = 0001_2$ and $g(2, 3) = 0011_2$. Therefore $X_g = \{0100_2, 0101_2, 0111_2, 1000_2, 1001_2, 1011_2, 1100_2, 1101_2, 1111_2\}$.

The set $X_f \cap X_g = \{0100_2, 0101_2, 1011_2, 1111_2\}$ is the set of all the consistent assignments of the occurrences such that each clause is satisfied.

We can formalize our observation as following.

Lemma 1. (★) There is a polynomial time reduction from a given instance of 3-CNF-SAT with n variables and m clauses into an instance of FAMILY INTERSECTION with $f : [n] \times [2] \rightarrow \mathbb{N}$ and $g : [m] \times [7] \rightarrow \mathbb{N}$ such that $\max X_f < 2^{3m}$ and $\max X_g < 2^{3m}$.

The proof is straightforward and its idea should be illustrated by the Example 1. It is moved to the Appendix due to space limitations.

	$\langle \hat{1}_{(1)}, \hat{1}_{(2)} \rangle$	$\langle \hat{1}_{(3)}, \hat{2} \rangle$	$\langle \hat{2}, \hat{1}_{(4)} \rangle$	$\langle \hat{2}, \hat{2} \rangle$
$\langle 1_{(i)}, 1_{(ii)} \rangle$	$f(\textcolor{red}{1}, 1_{(i)}) + f(\textcolor{blue}{2}, 1_{(ii)})$	$f(\textcolor{green}{3}, 1_{(i)})$	$f(\textcolor{blue}{4}, 1_{(ii)})$	0
$\langle 1_{(i)}, 2_{(ii)} \rangle$	$f(\textcolor{red}{1}, 1_{(i)}) + f(\textcolor{blue}{2}, 2_{(ii)})$	$f(\textcolor{green}{3}, 1_{(i)})$	$f(\textcolor{blue}{4}, 2_{(ii)})$	0
$\langle 2_{(i)}, 1_{(ii)} \rangle$	$f(\textcolor{red}{1}, 2_{(i)}) + f(\textcolor{blue}{2}, 1_{(ii)})$	$f(\textcolor{green}{3}, 2_{(i)})$	$f(\textcolor{blue}{4}, 1_{(ii)})$	0
$\langle 2_{(i)}, 2_{(ii)} \rangle$	$f(\textcolor{red}{1}, 2_{(i)}) + f(\textcolor{blue}{2}, 2_{(ii)})$	$f(\textcolor{green}{3}, 2_{(i)})$	$f(\textcolor{blue}{4}, 2_{(ii)})$	0

Figure 2: The weights on the edges of a graph encoding an f -family for $f : [4] \times [2] \rightarrow \mathbb{N}$. The lower indices (1), (2), (3) and (4) are added to indicate the correspondence between the occurrences of $\hat{1}$ and the elements of $[n]$ (the first argument of the function f). The lower indices (i) and (ii) are added to indicate the correspondence between the second argument of the function f and the position in the (two element) sequence $\langle \cdot, \cdot \rangle$.

2.2 From FAMILY INTERSECTION to COMMON MATCHING WEIGHT.

Consider an f -family X_f and a g -family X_g for some functions $f : [n] \times [2] \rightarrow \mathbb{N}$ and $g : [m] \times [7] \rightarrow \mathbb{N}$. In this section we show how to encode X_f in some weighted bipartite graph G_1 so that the set of the weights of the perfect matchings in G_1 will be equal to X_f . Similarly we will encode X_g in some bipartite graph G_2 such that the set of the weights of the perfect matchings in G_2 will be equal to X_g . So the set $X_f \cap X_g$ is nonempty if and only if G_1 and G_2 contain perfect matchings with the same weight. Moreover the number of the vertices of the graph G_1 will be $O\left(\frac{n}{\log n}\right)$ and the number of the vertices of the graph G_2 will be $O\left(\frac{m}{\log m}\right)$. This is a crucial step of our construction, because the instance size decreases (by a logarithmic factor).

Before we describe the reduction we need the following technical lemma which describe constructions of permutations of some specified properties. The permutations correspond naturally to perfect matchings in bipartite graphs. Elements of $[k]^b$ will be treated as b -character words over alphabet $[k]$, i.e. for $x \in [k]$ and $w \in [k]^b$ by xw we mean the word of length $b+1$ obtained by concatenating x and w . For convenience we define a set $\hat{\mathbb{N}} = \{\hat{0}, \hat{1}, \hat{2}, \dots\}$ as a copy of the natural numbers \mathbb{N} and for every $n \in \mathbb{N}$ we define $\hat{[n]} = \{\hat{1}, \hat{2}, \dots, \hat{n}\}$. Every set $\hat{[k]}^b$ is just a copy of $[k]^b$ so we refer to bijections between $[k]^b$ and $\hat{[k]}^b$ as to permutations.

Lemma 2. (★) *Let $b \in \mathbb{N}$ and $\alpha : \hat{[k]}^b \times [b] \rightarrow [k] \cup \{\perp\}$ such that for every $\hat{w} \in \hat{[k]}^b$ and for every $i \in [b]$ holds $\alpha(\hat{w}, i) \neq \perp$ if and only if $\hat{w}_i = \hat{1}$. There is a permutation $\phi : \hat{[k]}^b \rightarrow [k]^b$ such that for every $\hat{w} \in \hat{[k]}^b$ and for every $i \in [b]$ if $\hat{w}_i = \hat{1}$ then $\phi(\hat{w})_i = \alpha(\hat{w}, i)$.*

Now we can describe the reduction.

Lemma 3. *For a function $f : [n] \times [k] \rightarrow \mathbb{N}$ there is a full bipartite graph $G = (V_1 \cup V_2, E, w)$ such that*

- *for every $x \in X_f$ there exists a perfect matching M of G such that $w(M) = x$,*
- *for every perfect matching M of G we have $w(M) \in X_f$,*
- $|V_1| = |V_2| = O\left(\frac{nk^2 \log k}{\log n + \log k}\right)$.

Proof. Let us consider the smallest $b \in \mathbb{N}_+$ such that $c = b \cdot k^{b-1} \geq n$. Later we will show that $|V_1| = |V_2| = k^b$ is sufficient.

For convenience we extend our chosen function $f : [n] \times [k] \rightarrow \mathbb{N}$ to $f : [c] \times [k] \rightarrow \mathbb{N}$ in such a way that for every $i = n + 1, n + 2, \dots, c$ and for every $j \in [k]$ we put $f(i, j) = 0$. Note that the f -family X_f does not change after this extension.

Let $V_1 = [\hat{k}]^b$ and $V_2 = [k]^b$ be the sets of words of length b over the alphabets respectively $[\hat{k}]$ and $[k]$. Note that $|V_1| = |V_2| = k^b$.

Let $\beta : V_1 \times [b] \rightarrow [c] \cup \{\perp\}$ be any function such that if $\hat{w}_j \neq \hat{1}$ then $\beta(\hat{w}, j) = \perp$ and every value from the set $[c]$ is used exactly once, i.e., for every $x \in [c]$ there is exactly one argument $(\hat{w}, j) \in V_1 \times [b]$ such that $\beta(\hat{w}, j) = x$. Note that such a function always exists because the total number of the occurrences of $\hat{1}$ in all the words in V_1 is exactly $c = b \cdot k^{b-1}$.

Now we define our weight function $w : V_1 \times V_2 \rightarrow \mathbb{N}$ as follows

$$w(\hat{t}, u) = \sum_{\substack{i \in [b] \\ \beta(\hat{t}, i) \neq \perp}} f(\beta(\hat{t}, i), u_i).$$

An example of such weight function can be found in Figure 2 (or in the Appendix in Figure 6 as a picture of a bipartite graph).

Note that because β picks every value from the set $[c]$ exactly once then for every permutation $\phi : V_1 \rightarrow V_2$ we have

$$\sum_{\hat{t} \in V_1} w(\hat{t}, \phi(\hat{t})) = \sum_{\hat{t} \in V_1} \sum_{\substack{i \in [b] \\ \beta(\hat{t}, i) \neq \perp}} f(\beta(\hat{t}, i), \phi(\hat{t})_i) \in X_f.$$

In other words the set of the weights of all perfect matchings in G is a subset of X_f as required.

We also need to show that for every $x \in X_f$ there exists some permutation $\phi : V_1 \rightarrow V_2$ such that $\sum_{\hat{t} \in V_1} w(\hat{t}, \phi(\hat{t})) = x$. This permutation gives us a corresponding perfect matching of weight x in G .

Let us take a function $\sigma : [c] \rightarrow [k]$ such that $x = \sum_{i \in [c]} f(i, \sigma(i))$, which exists by the definition of X_f . Define $\alpha : [\hat{k}]^b \times [b] \rightarrow [k] \cup \{\perp\}$ as follows:

$$\alpha(\hat{u}, i) = \begin{cases} \sigma(\beta(\hat{u}, i)) & \text{for } \hat{u}_i = \hat{1} \\ \perp & \text{for } \hat{u}_i \neq \hat{1}. \end{cases}$$

Now we can use Lemma 2 with function α to obtain a permutation $\phi : [\hat{k}]^b \rightarrow [k]^b$ such that for every $\hat{u} \in [\hat{k}]^b$ and for every $i \in [b]$ if $\hat{u}_i = \hat{1}$ then $\phi(\hat{u})_i = \sigma(\beta(\hat{u}, i))$.

So we have that

$$\sum_{\hat{u} \in [\hat{k}]^b} w(\hat{u}, \phi(\hat{u})) = \sum_{\hat{u} \in [\hat{k}]^b} \sum_{\substack{i \in [b] \\ \beta(\hat{u}, i) \neq \perp}} f(\beta(\hat{u}, i), \sigma(\beta(\hat{u}, i))) = \sum_{i \in [c]} f(i, \sigma(i)) = x.$$

Hence we have shown that X_f is the set of weights of all perfect matchings in graph G . The last thing is to show that the number of the vertices is sufficiently small. We know that $(b-1) \cdot k^{b-2} < n$ so $(b-1) \cdot k^{b-1} < nk$ and then $b-1 < \log_k \frac{nk}{\log_k \frac{nk}{\log_k \frac{nk}{\dots}}}$. Therefore $k^b < k \cdot \frac{nk}{\log_k \frac{nk}{\log_k \frac{nk}{\dots}}} = O\left(\frac{nk^2}{\log_k nk}\right) = O\left(\frac{nk^2 \log k}{\log n + \log k}\right)$. So $|V_1| = |V_2| = k^b = O\left(\frac{nk^2 \log k}{\log n + \log k}\right)$, as required. \square

Lemma 3 immediately implies the following result.

Lemma 4. *There is a polynomial time reduction that for an instance $I = (f, g)$ of FAMILY INTERSECTION with $f : [a] \times [b] \rightarrow \mathbb{N}$ and $g : [c] \times [d] \rightarrow \mathbb{N}$ reduces it into an instance of COMMON MATCHING WEIGHT $J = (G_1, G_2)$ with $|V(G_1)| = O\left(\frac{ab^2 \log b}{\log a + \log b}\right)$ and $|V(G_2)| = O\left(\frac{cd^2 \log d}{\log c + \log d}\right)$ vertices. The sets of the weights of all perfect matchings in G_1 and in G_2 are equal respectively to X_f and X_g .*

Together with Lemma 1 we obtain the following theorem.

Theorem 5. *There is a polynomial time reduction from a given instance of 3-CNF-SAT with n variables and m clauses into an instance of COMMON MATCHING WEIGHT with $|V(G_1)| = O\left(\frac{n}{\log n}\right)$, $|V(G_2)| = O\left(\frac{m}{\log m}\right)$ and the maximum matching weights bounded by 2^{3m} .*

A commonly know corollary of the Sparsification Lemma from [4] is:

Corollary 6. *There is no algorithm solving 3-CNF-SAT in $2^{o(n+m)}$ -time where n is the number of variables and m is the number of clauses unless ETH fails.*

Using Corollary 6 we can prove the following lower bound.

Corollary 7. *There is no algorithm solving COMMON MATCHING WEIGHT in $2^{o(n \log n)} \text{poly}(r)$ -time where n is the total number of vertices, and r is the bit size of the input, unless ETH fails.*

Proof. For a given instance of 3-CNF-SAT with n variables and m clauses we can use the reduction from Theorem 5 to obtain an instance of CHANNEL ASSIGNMENT. The total number of the vertices in the new instance is

$$|V(G_1)| + |V(G_2)| = O\left(\frac{n}{\log n} + \frac{m}{\log m}\right) = O\left(\frac{n+m}{\log(n+m)} + \frac{n+m}{\log(n+m)}\right) = O\left(\frac{n+m}{\log(n+m)}\right)$$

because the function $\frac{n}{\log n}$ is nondecreasing for the sufficiently big values of n . Weights of the matchings are bounded by 2^{3m} and the bit size of the instance

$$r = O\left(\left(\frac{n+m}{\log(n+m)}\right)^2 \log 2^{3m}\right) = \text{poly}(nm).$$

Then let us assume that there is an algorithm solving COMMON MATCHING WEIGHT in $2^{o(n \log n)} \text{poly}(r)$ -time. Then we could solve our instance in time

$$2^{O\left(\frac{n+m}{\log(n+m)} \log\left(\frac{n+m}{\log(n+m)}\right)\right)} \text{poly}(\text{poly}(nm)) = 2^{O\left(\frac{n+m}{\log(n+m)} \log(n+m)\right)} \text{poly}(nm) = 2^{o(n+m)}$$

which contradicts ETH by Corollary 6. □

3 Hardness of CHANNEL ASSIGNMENT

Consider two weighted full bipartite graphs G_1 and G_2 . We would like to encode them in a CHANNEL ASSIGNMENT instance in such a way that this CHANNEL ASSIGNMENT instance is a YES-instance if and only if there are two perfect matchings, one in G_1 and the other in G_2 , of the same weight.

Consider an instance $I = (V, d, s)$ of CHANNEL ASSIGNMENT. We say that $c : V \rightarrow \mathbb{Z}$ is a YES-coloring if c is a proper coloring and has span at most s . Note that an instance of CHANNEL ASSIGNMENT is a YES-instance if and only if it has a YES-coloring.

Our approach is that we encode those graphs G_1 and G_2 separately in such a way that we have a special vertex v_M whose color in every YES-coloring represents a weight of some perfect matching in G_1 and on the other hand in every YES-coloring its color represents (in a similar way) a weight of some perfect matching in G_2 . So a YES-coloring coloring would be possible if and only if the graphs G_1 and G_2 have two perfect matchings, one in G_1 and the other in G_2 , with equal weights.

Before we present a way to encode a weighted full bipartite graph in a CHANNEL ASSIGNMENT instance we would like to present the two lemmas to merge those two encoded graphs into a one instance of CHANNEL ASSIGNMENT. In order to do that we use the following concepts.

We say that instance I is (x, y) -spanned for some vertices $x, y \in V$ if for every YES-coloring c of I we have $|c(x) - c(y)| = s - 1$.

We say that an instance $I = (V, d, s)$ of CHANNEL ASSIGNMENT is (X, Y) -spanned for some nonempty subsets of the vertices $\emptyset \neq X, Y \subseteq V$ if it is (x, y) -spanned for every two vertices $x \in X$ and $y \in Y$.

Lemma 8. (★) *For every (u, v) -spanned instance $I_1 = (V_1, d_1, s)$ and (w, z) -spanned instance $I_2 = (V_2, d_2, s)$ of CHANNEL ASSIGNMENT there is a $(\{u, w\}, \{v, z\})$ -spanned instance $I = (V_1 \cup V_2, d, s)$ of CHANNEL ASSIGNMENT such that*

- (i) *for every YES-coloring c of I the coloring $c|_{V_1}$ is a YES-coloring of I_1 and the coloring $c|_{V_2}$ is a YES-coloring of I_2 ,*
- (ii) *for every YES-coloring c_1 of I_1 and every YES-coloring c_2 of I_2 such that $c_1(u) = c_2(w)$, $c_1(v) = c_2(z)$ and for every $x \in V_1 \cap V_2$ we have $c_1(x) = c_2(x)$ there exists a YES-coloring c of I such that $c|_{V_1} = c_1$ and $c|_{V_2} = c_2$.*

Lemma 9. (★) *For every (v_L, v_R) -spanned instance $I = (V, d, s)$ of CHANNEL ASSIGNMENT and for every numbers $l, r \in \mathbb{N}$ there exists a (w_L, w_R) -spanned instance $I' = (V \cup \{w_L, w_R\}, d', l + s + r)$ such that*

- (i) *for every YES-coloring c of I there is a YES-coloring c' of I' such that $c'|_V = c$,*
- (ii) *for every YES-coloring c' of I' such that $c'(w_L) \leq c'(w_R)$ we have that*
 - *a coloring $c'|_V$ is a YES-coloring of I ,*
 - *$c'(v_L) = c'(w_L) + l$ and $c'(v_R) = c'(w_R) - r$.*

The proofs of these two lemmas are straightforward. They are moved to the Appendix due to space limitations.

Lemma 10. *Let $G = (V_1 \cup V_2, E, w)$ be a weighted full bipartite graph with nonnegative weights and such that $|V_1| = |V_2|$. Let $n = |V_1|$, $m = \max_{e \in E} w(e)$, $M = n \cdot m + 1$, $l = (4n - 1) \cdot M$ and $s = (8n - 1) \cdot M$. There exists a (v_L, v_R) -spanned instance $I = (V, d, s)$ of CHANNEL ASSIGNMENT with $|V| = O(n)$ and such that for some vertex $v_M \in V$,*

- (i) *for every YES-coloring c of I such that $c(v_L) \leq c(v_R)$ there exists a perfect matching M_G in G such that $c(v_M) = c(v_L) + l + w(M_G)$ and*
- (ii) *for every perfect matching M_G in G there exists a YES-coloring c of I such that $c(v_L) \leq c(v_R)$ and $c(v_M) = c(v_L) + l + w(M_G)$.*

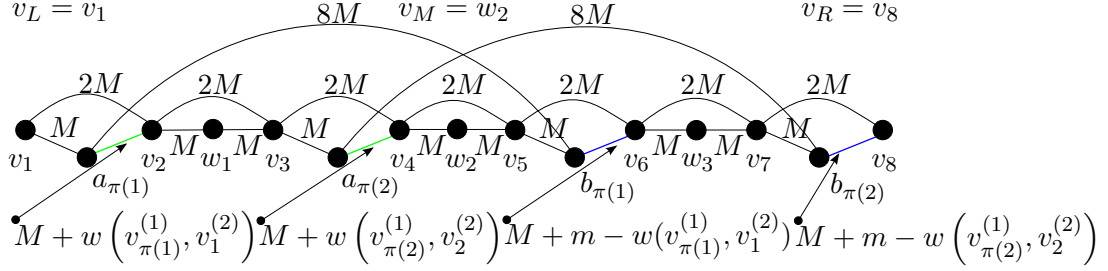


Figure 3: A weighted full bipartite graph (G, w) with $|V_1| = |V_2| = 2$ encoded in a CHANNEL ASSIGNMENT form. The color of the vertex $v_M = w_2$ corresponds to the weight of the perfect matching in G given by the permutation π and is equal to $c(v_M) = c(v_L) + 7M + w(M_\pi)$. The picture is simplified. Some of the edges and corresponding to them minimum distances are omitted in the picture.

Proof. Let $V_1 = \{v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)}\}$. We will build our CHANNEL ASSIGNMENT instance step by step. A simplified picture of the instance can be found in Figure 3.

Let us introduce the vertices $v_L = v_1, v_2, \dots, v_{4n} = v_R$ to set V . Because of the symmetry we can assume that for every coloring c of our instance we have $c(v_L) \leq c(v_R)$.

We set the minimum distance $d(v_L, v_R) = s - 1$. Then for every YES-coloring c we have that $|c(v_L) - c(v_R)| = s - 1$ so our instance is (v_L, v_R) -spanned.

For every $i, j \in [4n]$ such that $i \neq j$ and $\{i, j\} \neq \{1, 4n\}$ we set the minimum distance $d(v_i, v_j) = |i - j| \cdot 2M$. Then we can prove the following claim.

Claim 1 For every YES-coloring c and for every $i < j$ we have that $c(v_i) < c(v_j)$.

Proof of the claim: Indeed if the colors $c(v_1), c(v_2), \dots, c(v_{4n})$ are not strictly increasing or strictly decreasing then the distances between the consecutive colors of the set $c(\{v_1, v_2, \dots, v_{4n}\})$ are at least $2M$ and at least one of them is at least $4M$. So it would need to use at least $(4n - 2) \cdot 2M + 4M + 1 > (8n - 1) \cdot M = s$ colors. In addition we have assumed that for all colorings $c(v_L) \leq c(v_R)$ so the colors are strictly increasing. This proves the claim.

Note that for every YES-coloring and for every $i \in [4n - 1]$ we have

$$2M \leq c(v_{i+1}) - c(v_i) \leq 2M + n \cdot m < 3M, \quad (1)$$

for otherwise $c(v_R) - c(v_L) \geq (4n - 2) \cdot 2M + 2M + n \cdot m + 1 = s$, so c has the span at least $s + 1$, a contradiction.

Let us introduce new vertices $w_1, w_2, \dots, w_{2n-1}$ to set V . For every $i \in [2n - 1]$ and $j \in [4n]$ we set the minimum distances $d(w_i, v_j) = |4i + 1 - 2j| \cdot M$. For every YES-coloring c and for every $i \in [2n - 1]$ we have $c(v_{2i}) + M \leq c(w_i) \leq c(v_{2i+1}) - M$ by (1), for otherwise we have that $c(v_j) \leq c(w_i) \leq c(v_{j+1})$ for some $j \neq 2i$ (because $c(v_{4n}) - c(v_1) = s - 1$ so every YES-coloring uses only colors from the interval $[c(v_1), c(v_{4n})]$) and then $c(v_{j+1}) - c(v_j) \geq d(v_j, w_i) + d(w_i, v_{j+1})$ and $\{v_j, v_{j+1}\} \neq \{v_{2i}, v_{2i+1}\}$ so at least one of these two distances is at least $3M$ and therefore $c(v_{j+1}) - c(v_j) \geq 3M + M > 3M$, a contradiction with (1). Thus infer the following claim.

Claim 2 For every YES-coloring c the colors of the vertices in the sequence

$$v_1, v_2, w_1, v_3, v_4, w_2, v_5, \dots, v_{4n-2}, w_{2n-1}, v_{4n-1}, v_{4n} \quad (2)$$

are increasing.

We introduce new vertices a_1, a_2, \dots, a_n and for every $i \in [n]$ and $j \in [4n]$ we set the minimum distances

$$d(a_i, v_j) = \begin{cases} M + w(v_i^{(1)}, v_{j/2}^{(2)}) & \text{when } j \leq 2n \text{ and } 2 \mid j \\ M & \text{when } j \leq 2n \text{ and } 2 \nmid j \\ (j - 2n) \cdot 2M + M & \text{when } j > 2n. \end{cases}$$

Then for every YES-coloring c and for every $i \in [n]$ we have $c(a_i) \leq c(v_{2n})$ because in other case we have $c(v_j) \leq c(a_i) \leq c(v_{j+1})$ for some $j \geq 2n$ and then $c(v_{j+1}) - c(v_j) \geq d(v_j, a_i) + d(a_i, v_{j+1}) \geq M + 3M > 3M$, a contradiction with (1).

Moreover for every $i \in [n]$ and every $j \in [2n-1]$ we set the minimum distance $d(a_i, w_j) = 2M$. Therefore by (1) and (2) for every YES-coloring c and every $i \in [n]$ the vertex a_i is colored with the color from one of the intervals $(c(v_{2j-1}), c(v_{2j}))$ for some $j \in [n]$.

Finally for every $i, j \in [n]$ such that $i \neq j$ we set the minimum distance $d(a_i, a_j) = 4M$ so by (1) we know that for every YES-coloring c and every $i \in [n]$ exactly one vertex a_j of the vertices a_1, a_2, \dots, a_n is colored with the color from the interval $(c(v_{2i-1}), c(v_{2i}))$. The assignment of vertices a_1, a_2, \dots, a_n to intervals $(c(v_1), c(v_2)), (c(v_3), c(v_4)), \dots, (c(v_{2n-1}), c(v_{2n}))$ determines a permutation $\pi_c : [n] \rightarrow [n]$, i.e., $\pi_c(i) = j$ if a_j gets a color from $(c(v_{2i-1}), c(v_{2i}))$. Hence we get the following claim:

Claim 3 For every YES-coloring c there is a permutation π_c such that the colors of the vertices of the sequence

$$v_1, a_{\pi_c(1)}, v_2, w_1, v_3, a_{\pi_c(2)}, v_4, w_2, v_5, \dots, v_{2n-1}, a_{\pi_c(n)}, v_{2n}$$

are increasing.

Similarly we introduce new vertices b_1, b_2, \dots, b_n and for every $i \in [n]$ and $j \in [4n]$ we set the minimum distances

$$d(b_i, v_j) = \begin{cases} (2n - j + 1) \cdot 2M + M & \text{when } j \leq 2n \\ M + m - w \left(v_i^{(1)}, v_{j/2-n}^{(2)} \right) & \text{when } j > 2n \text{ and } 2 \mid j \\ M & \text{when } j > 2n \text{ and } 2 \nmid j. \end{cases}$$

Also for every $i \in [n]$ and every $j \in [2n-1]$ we set the minimum distance $d(b_i, w_j) = 2M$ and for every $i, j \in [n]$ such that $i \neq j$ we set the minimum distance $d(b_i, b_j) = 4M$. Hence similarly as before, for every YES-coloring c and every $i \in [n]$ exactly one vertex b_j of the vertices b_1, b_2, \dots, b_n is colored with the color from the interval $(c(v_{2n+2i-1}), c(v_{2n+2i}))$. Analogously as before, the colors of the vertices b_1, b_2, \dots, b_n determine a permutation $\rho_c : [n] \rightarrow [n]$. Thus we have the following claim.

Claim 4 For every YES-coloring c there is a permutation ρ_c such that the colors of the vertices in the sequence

$$v_{2n+1}, b_{\rho_c(1)}, v_{2n+2}, w_{n+1}, v_{2n+3}, b_{\rho_c(2)}, v_{2n+4}, w_{n+2}, v_{2n+5}, \dots, v_{4n-1}, b_{\rho_c(n)}, v_{4n}$$

are increasing.

For every $i \in [n]$ we set the minimum distance $d(a_i, b_i) = n \cdot 4M$. Then we know that for every YES-coloring c we have $\pi_c^{-1}(i) \leq \rho_c^{-1}(i)$ for otherwise we can take $j = 2\pi_c^{-1}(i) - 1$ and $k = 2n + 2\rho_c^{-1}(i)$ and then $(c(b_i) - c(a_i)) + 2M \leq c(v_j) - c(v_k)$ and $k - j \leq 2n$ so the sequence $v_1, v_2, \dots, v_j, v_k, \dots, v_{4n}$ has at least $4n - 2n + 1 = 2n + 1$ elements so $c(v_{4n}) - c(v_1) \geq (2n - 1) \cdot 2M + (c(v_k) - c(v_j)) \geq (2n - 1) \cdot 2M + (c(b_i) - c(a_i)) + 2M \geq (2n - 1) \cdot 2M + n \cdot 4M + 2M = n \cdot 8M > (n - 1) \cdot 8M - 1 = s - 1$, a contradiction. Since π_c and ρ_c are permutations, we further infer that for every YES-coloring c we have $\pi_c = \rho_c$. Hence we have the following claim.

Claim 5 For every YES-coloring c there is a permutation π_c such that the colors of the vertices in the sequence

$$v_1, a_{\pi_c(1)}, v_2, w_1, v_3, a_{\pi_c(2)}, v_4, w_2, v_5, \dots, v_{2n-1}, a_{\pi_c(n)}, v_{2n}, w_n, v_{2n+1}, b_{\pi_c(1)}, v_{2n+2}, w_{n+1}, v_{2n+3}, b_{\pi_c(2)}, v_{2n+4}, w_{n+2}, v_{2n+5}, \dots, v_{4n-1}, b_{\pi_c(n)}, v_{4n}$$

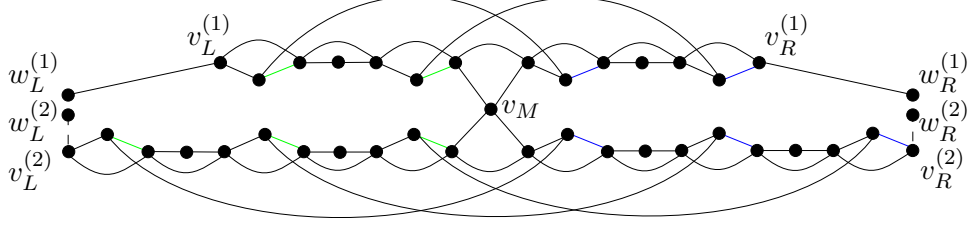


Figure 4: Two weighted full bipartite graphs (G_1, w_1) (with $n_1 = 2$) and (G_2, w_2) (with $n_2 = 3$) encoded in a CHANNEL ASSIGNMENT form. The color of the vertex $v_M = w_2$ corresponds to the weight of some perfect matching in G_1 and to the weight of some perfect matching in G_2 . These two weights have to be equal. The picture is simplified. Some of the edges are omitted in the picture. Note that the values M and m can be different for (G_1, w_1) and for (G_2, w_2) .

are increasing.

This ends the description of the instance I . Note that I is (v_L, v_R) -spanned because $d(v_L, v_R) = s - 1$. Let us put $v_M = w_n$. We are going to show the following claim.

Claim 6 (★) Let $\pi : [n] \rightarrow [n]$ be any permutation and $M_\pi = \{v_i^{(1)} v_{\pi(i)}^{(2)} : i \in [n]\}$ be the corresponding perfect matching in G . There is exactly one YES-coloring c such that $\pi_c = \pi$. Moreover $c(v_M) = c(v_L) + l + w(M_\pi)$.

To prove the claim it is sufficient to check all the introduced minimum allowed distances d and the span for the coloring implied by the sequence as in Claim 5. This is a simple manual check but due to its length the proof of the claim is moved to the Appendix.

Thus there is a one-to-one correspondence between permutations and YES-colorings. Moreover we know that for every YES-coloring c we have $c(v_M) = c(v_L) + l + w(M_{\pi_c})$ where M_{π_c} is the perfect matching in G corresponding to permutation π_c . Hence we have shown (i) and (ii) as required. \square

Lemma 11. (★) *There is a polynomial time reduction such that for a given instance $I = (G_1, G_2)$ of COMMON MATCHING WEIGHT with $n_1 = |V(G_1)|$, $n_2 = |V(G_2)|$ and such that the weight functions of G_1 and G_2 are bounded by respectively m_1 and m_2 reduces it into an instance of CHANNEL ASSIGNMENT with $O(n_1 + n_2)$ vertices and the maximum edge weight in $O(n_1^2 m_1 + n_2^2 m_2)$.*

In the proof we use Lemma 10 to encode G_1 and G_2 in two instances of CHANNEL ASSIGNMENT, then we extend them to the common length using Lemma 9 and finally we merge them using Lemma 8. The proof is straightforward and is moved to the Appendix due to space limitations. The simplified picture of the obtained CHANNEL ASSIGNMENT instance can be found in Figure 4.

Now we can use the results of Section 2 to get following two corollaries.

Corollary 12. *There is no algorithm solving CHANNEL ASSIGNMENT in $2^{o(n \log n)} \text{poly}(r)$ where n is the number of the vertices and r is the bit size of the instance unless ETH fails.*

Proof. For a given instance of COMMON MATCHING WEIGHT with n vertices and the weights bounded by m we can transform it by Lemma 11 into an instance of CHANNEL ASSIGNMENT with $n' = O(n)$ vertices and the weights bounded by $\ell = O(n^2 m)$. Note that for the bit size r' of the new instance we have $\text{poly}(r') = \text{poly}((n')^2 \ell) = \text{poly}(n, m) = \text{poly}(r)$.

Let us assume that we can solve CHANNEL ASSIGNMENT in $2^{o(n \log n)} \text{poly}(r)$ -time. Then we can solve our instance in time $2^{o(n' \log n')} \text{poly}(r') = 2^{o(n \log n)} \text{poly}(r)$ which contradicts ETH by Corollary 7. \square

Corollary 13. (★) *There is no algorithm solving CHANNEL ASSIGNMENT in $2^{n \cdot o(\log \log \ell)} \text{poly}(r)$ where n is the number of the vertices and r is the bit size of the instance unless ETH fails.*

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A Omitted Proofs from Section 2

A.1 Omitted Proofs from Section 2.1

Lemma 14 (Lemma 1). *There is a polynomial time reduction from a given instance of 3-CNF-SAT with n variables and m clauses into an instance of FAMILY INTERSECTION with $f : [n] \times [2] \rightarrow \mathbb{N}$ and $g : [m] \times [7] \rightarrow \mathbb{N}$ such that $\max X_f < 2^{3m}$ and $\max X_g < 2^{3m}$.*

Proof. Let $V = \{v_1, v_2, \dots, v_n\}$ and $C = \{c_1, c_2, \dots, c_m\}$ be the sets of variables and clauses of the input formula, respectively.

Let $D = \{d_1, d_2, \dots, d_{3m}\}$ be the set of all $3m$ occurrences of our n variables in our m clauses. We will treat these occurrences as separate variables.

For every variable $v_i \in V$ we define a set $I_i \subseteq [3m]$ such that $j \in I_i$ if and only if d_j is an occurrence of the variable v_i .

Similarly for every clause $c_i \in C$ we define a set $J_i \subseteq [3m]$ such that $j \in J_i$ if and only if d_j an occurrence (of any variable) belonging to the clause c_i . For every $i \in [m]$ we have $|J_i| = 3$.

For every clause c_i we can treat the subsets of J_i as the assignments of the occurrences d_j belonging to the clause c_i . We treat the subset $K \subseteq J_i$ as the assignment of the occurrences in the clause c_i such that the occurrence d_j is set to 1 if and only if $j \in K$, otherwise it is set to 0. We say that $K \subseteq J_i$ satisfies the clause c_i if the corresponding assignment of the occurrences satisfies this clause.

For every clause $c_i \in C$ let us define the set $P_i = \{K \subseteq J_i : K \text{ satisfies the clause } c_i\}$. Again note that we treat here all the occurrences as the different variables. Note that $|P_i| = 7$ for every i , so we can denote $P_i = \{P_i^1, P_i^2, \dots, P_i^7\}$.

A number from $0, 1, \dots, 2^{3m} - 1$ can be interpreted in the binary system as the characteristic vector of length $3m$ of a subset of the indices of the occurrences i.e., that the i -th bit represents if the occurrence d_i belongs to this subset or not.

We define a function $f : [n] \times [2] \rightarrow \mathbb{N}$ such that for every $i \in [n]$ we set $f(i, 1) = \sum_{j \in I_i} 2^{j-1}$ and $f(i, 2) = 0$. In other words the number $f(i, 1)$ represents the characteristic vector of all the occurrences of the variable v_i .

Note that $\{\sigma : [n] \rightarrow [2]\}$ corresponds to the set of all assignments of variables. Therefore X_f is the set of all the characteristic vectors which represent all the assignments of the occurrences such that all the occurrences of the same variable have the same value.

We define a function $g : [m] \times [7] \rightarrow \mathbb{N}$ such that for every $i \in [m]$ and for every $j \in [7]$ we can set $g(i, j) = \sum_{K \in P_i^j} 2^{k-1}$. Then for every $i \in [m]$ the numbers $g(i, 1), g(i, 2), \dots, g(i, 7)$ represent the characteristic vectors of all the assignments of the occurrences in the clause c_i which satisfy this clause.

Therefore the set X_g is the set of all the characteristic vectors which represents the assignments of all $3m$ occurrences such that all the clauses are satisfied.

It follows that the set $X_f \cap X_g$ is the set of all the characteristic vectors which represent the assignments of the occurrences such that all the occurrences of the same variable have the same value and all the clauses are satisfied. In other words, elements of $X_f \cap X_g$ correspond to satisfying assignments. \square

A.2 Omitted Proofs from Section 2.2

The first lemma provides a way of merging k permutations $\phi_1, \phi_2, \dots, \phi_k : [\hat{k}]^b \rightarrow [k]^b$ into one permutation $\phi : [\hat{k}]^{b+1} \rightarrow [k]^{b+1}$ in a way specified by a function $\rho : [\hat{k}]^b \rightarrow [k]$.

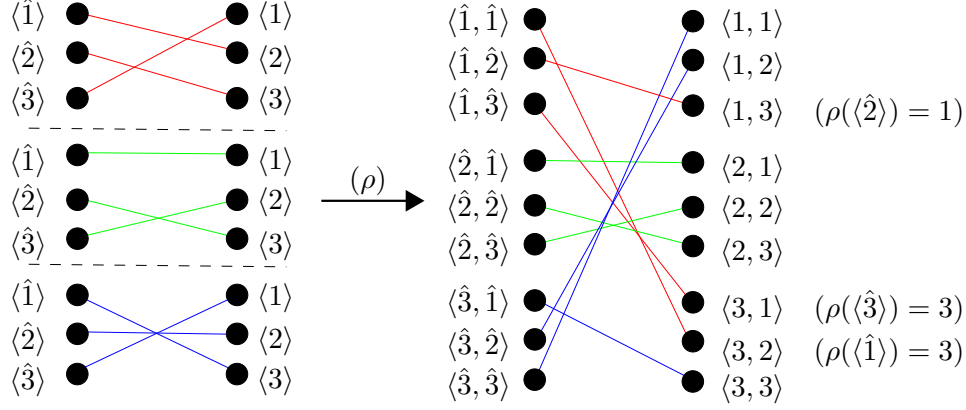


Figure 5: Merging three permutations (presented as perfect matchings) with respect to the function ρ such that $\rho(\langle \hat{1} \rangle) = 3$, $\rho(\langle \hat{2} \rangle) = 1$ and $\rho(\langle \hat{3} \rangle) = 3$.

Lemma 15. For every $b \in \mathbb{N}$ and for a given sequence of permutations $\phi_1, \phi_2, \dots, \phi_k : [\hat{k}]^b \rightarrow [k]^b$ and for every function $\rho : [\hat{k}]^b \rightarrow [k]$ there is a permutation $\phi : [\hat{k}]^{b+1} \rightarrow [k]^{b+1}$ such that

- (i) for every $x \in [k]$ and for every $\hat{w} \in [\hat{k}]^b$ there exists $y \in [k]$ such that $\phi(\hat{x}\hat{w}) = y\phi_x(\hat{w})$ and moreover
- (ii) for every $\hat{w} \in [\hat{k}]^b$ we have $\phi(\hat{1}\hat{w}) = \rho(\hat{w})\phi_1(\hat{w})$.

Before we proceed to the proof we suggest the reader to take a look at an example in Figure 5 ($b = 1, k = 3$).

Proof. We start with a permutation $\hat{x}\hat{w} \mapsto x\phi_x(\hat{w})$ which already satisfies the condition $\exists_y \phi(\hat{x}\hat{w}) = y\phi_x(\hat{w})$. Then we are going to swap the values for some (disjoint) pairs of the arguments in order to fulfill the condition $\phi(\hat{1}\hat{w}) = \rho(\hat{w})\phi_1(\hat{w})$. Such swaps are preserving the condition of being permutation. Moreover we perform only such swaps that preserve also the $\exists_y \phi(\hat{x}\hat{w}) = y\phi_x(\hat{w})$ condition.

For every $\hat{w} \in [\hat{k}]^b$ we need to put $\phi(\hat{1}\hat{w}) = \rho(\hat{w})\phi_1(\hat{w})$. Let us assign $x = \rho(\hat{w})$ and $\hat{u} = \phi_x^{-1}(\phi_1(\hat{w}))$. Note that $\phi_x(\hat{u}) = \phi_1(\hat{w})$. If $x \neq 1$ then to avoid a collision $\phi(\hat{1}\hat{w}) = \phi(\hat{x}\hat{u})$ we can put $\phi(\hat{x}\hat{u}) = 1\phi_1(\hat{w})$. So we have swapped the values for the arguments $\hat{1}\hat{w}$ and $\hat{x}\hat{u}$. Our function is still a permutation. Note that the condition $\exists_y \phi(\hat{x}\hat{u}) = y\phi_x(\hat{u})$ is still preserved because $\phi_x(\hat{u}) = \phi_1(\hat{w})$. We just need to show that the swaps can be performed independently.

For every $i \in [k]$ a function $\phi_i^{-1} \circ \phi_1$ is a permutation so for every $\hat{w} \in [\hat{k}]^b$ the values of $\rho(\hat{w})\phi_{\rho(\hat{w})}^{-1}(\phi_1(\hat{w}))$ are pairwise different. Indeed for two different $\hat{u}, \hat{w} \in [\hat{k}]^b$ either the values $\rho(\hat{u})$ and $\rho(\hat{w})$ are different or $\rho(\hat{u}) = \rho(\hat{w}) = x$ for some $x \in [k]$ and then $(\phi_x^{-1} \circ \phi_1)(\hat{u}) \neq (\phi_x^{-1} \circ \phi_1)(\hat{w})$ so then the values $\phi_{\rho(\hat{u})}^{-1}(\phi_1(\hat{u}))$ and $\phi_{\rho(\hat{w})}^{-1}(\phi_1(\hat{w}))$ are different. Therefore our pairs of the arguments to swap are pairwise disjoint. Thus all the swaps can be performed independently.

So for every $x \in [k]$ and $\hat{w} \in [\hat{k}]^b$ we have

$$\phi(\hat{x}\hat{w}) = \begin{cases} \rho(\hat{w})\phi_1(\hat{w}) & \text{for } \hat{x} = \hat{1} \\ 1\phi_x(\hat{w}) & \text{for } \hat{x} \neq \hat{1} \wedge \rho(\phi_1^{-1}(\phi_x(\hat{w}))) = x \\ x\phi_x(\hat{w}) & \text{in other cases.} \end{cases}$$

□

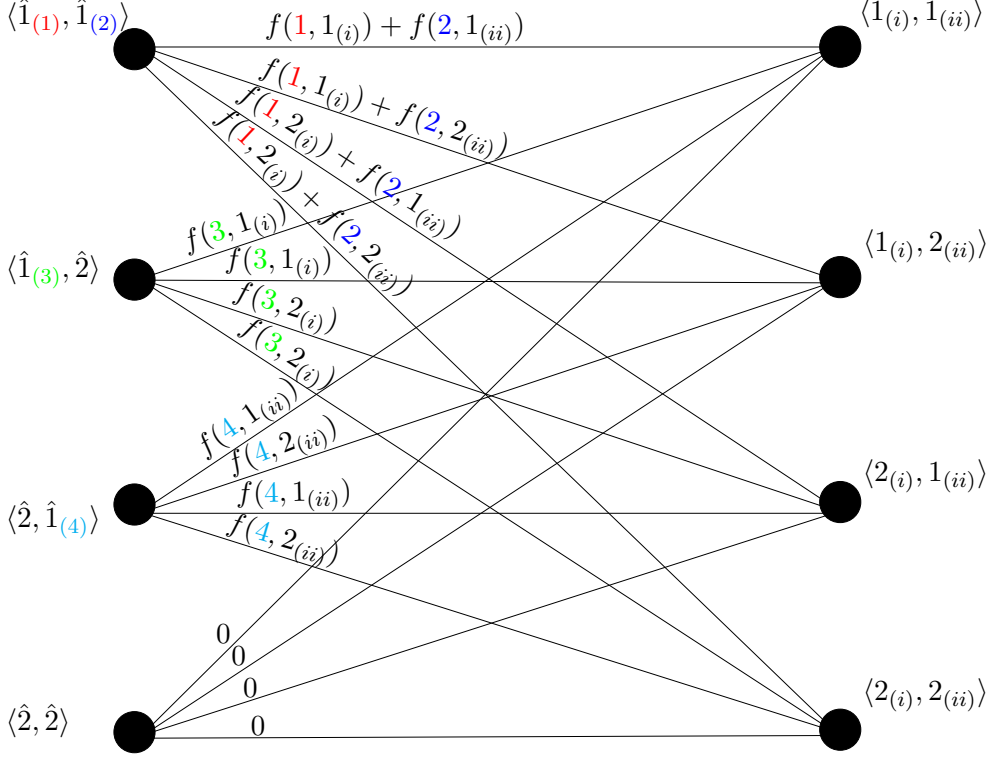


Figure 6: The graph encoding an f -family for $f : [4] \times [2] \rightarrow \mathbb{N}$. The lower indices (1), (2), (3) and (4) are added to indicate the correspondence between the occurrences of $\hat{1}$ and the elements of $[n]$ (the first argument of the function f). The lower indices (i) and (ii) are added to indicate the correspondence between the second argument of the function f and the position in the (two element) sequence $\langle \cdot, \cdot \rangle$.

Lemma 16 (Lemma 2). *Let $b \in \mathbb{N}$ and $\alpha : [\hat{k}]^b \times [b] \rightarrow [k] \cup \{\perp\}$ such that for every $\hat{w} \in [\hat{k}]^b$ and for every $i \in [b]$ holds $\alpha(\hat{w}, i) \neq \perp$ if and only if $\hat{w}_i = \hat{1}$. There is a permutation $\phi : [\hat{k}]^b \rightarrow [k]^b$ such that for every $\hat{w} \in [\hat{k}]^b$ and for every $i \in [b]$ if $\hat{w}_i = \hat{1}$ then $\phi(\hat{w})_i = \alpha(\hat{w}, i)$.*

Proof. We will use an induction on b .

For $b = 0$ we have $\phi(\varepsilon) = \varepsilon$.

For $b > 0$ we can define functions $\alpha_1, \alpha_2, \dots, \alpha_k : [\hat{k}]^{b-1} \times [b-1] \rightarrow [k] \cup \{\perp\}$ such that for every $x \in [k]$ every $\hat{w} \in [\hat{k}]^{b-1}$ and every $i \in [b-1]$ we put $\alpha_x(\hat{w}, i) = \alpha(\hat{x}\hat{w}, i+1)$.

From the inductive hypothesis for $b-1$ used for every function of $\alpha_1, \alpha_2, \dots, \alpha_k$ we got the permutations $\phi_1, \phi_2, \dots, \phi_k : [\hat{k}]^{b-1} \rightarrow [k]^{b-1}$ such that for every $x \in [k]$ for every $\hat{w} \in [\hat{k}]^{b-1}$ and for every $i \in [b-1]$ we have that if $\hat{w}_i = \hat{1}$ then $\phi_x(\hat{w})_i = \alpha_x(\hat{w}, i) = \alpha(\hat{x}\hat{w}, i+1)$.

Now we can use Lemma 15 to merge the permutations $\phi_1, \phi_2, \dots, \phi_k$ using a function $\rho : [\hat{k}]^{b-1} \rightarrow [k]$ such that $\rho(\hat{w}) = \alpha(\hat{1}\hat{w}, 1)$ for every $\hat{w} \in [\hat{k}]^{b-1}$. We obtain one permutation $\phi : [\hat{x}]^b \rightarrow [x]^b$ such that by Lemma 15 (i) for every $\hat{x} \in [\hat{k}]$, for every $\hat{w} \in [\hat{k}]^{b-1}$ and for every $i \in [b-1]$ we have that $\phi(\hat{x}\hat{w})_{i+1} = \phi_x(\hat{w})_i$ so if $\hat{w}_i = \hat{1}$ then $\phi(\hat{x}\hat{w})_{i+1} = \phi_x(\hat{w})_i = \alpha(\hat{x}\hat{w}, i+1)$. Also by Lemma 15 (ii), for every $\hat{w} \in [\hat{k}]^{b-1}$ we have that $\phi(\hat{1}\hat{w})_1 = \rho(\hat{w}) = \alpha(\hat{1}\hat{w}, 1)$. So for every $\hat{w} \in [\hat{k}]^b$ and for every $i \in [b]$ we have that if $\hat{w}_i = \hat{1}$ then $\phi(\hat{w})_i = \alpha(\hat{w}, i)$, as required. \square

B Omitted Proofs from Section 3

Lemma 17 (Lemma 8). *For every (u, v) -spanned instance $I_1 = (V_1, d_1, s)$ and (w, z) -spanned instance $I_2 = (V_2, d_2, s)$ of CHANNEL ASSIGNMENT there is a $(\{u, w\}, \{v, z\})$ -spanned instance $I = (V_1 \cup V_2, d, s)$ of CHANNEL ASSIGNMENT such that*

- (i) *for every YES-coloring c of I the coloring $c|_{V_1}$ is a YES-coloring of I_1 and the coloring $c|_{V_2}$ is a YES-coloring of I_2 ,*
- (ii) *for every YES-coloring c_1 of I_1 and every YES-coloring c_2 of I_2 such that $c_1(u) = c_2(w)$, $c_1(v) = c_2(z)$ and for every $x \in V_1 \cap V_2$ we have $c_1(x) = c_2(x)$ there exists a YES-coloring c of I such that $c|_{V_1} = c_1$ and $c|_{V_2} = c_2$.*

Proof. Let $B = \{u, w\} \times \{v, z\} \cup \{v, z\} \times \{u, w\}$ and let

$$d(x, y) = \begin{cases} s - 1 & \text{if } (x, y) \in B \\ \max\{d_1(x, y), d_2(x, y)\} & \text{if } x, y \in V_1 \cap V_2 \\ d_1(x, y) & \text{if } x, y \in V_1 \text{ and } x, y \notin V_1 \cap V_2 \\ d_2(x, y) & \text{if } x, y \in V_2 \text{ and } x, y \notin V_1 \cap V_2 \\ 0 & \text{otherwise.} \end{cases}$$

Our instance is $(\{u, w\}, \{v, z\})$ -spanned because for all the pairs in B we set the minimum allowed distance to at least $s - 1$.

Note that for $i = 1, 2$, for every $x, y \in V_i$ we have $d(x, y) \geq d_i(x, y)$. Hence every proper coloring c of I has the property that $c|_{V_1}$ is a proper coloring of I_1 and $c|_{V_2}$ is a proper coloring of I_2 . Also the maximum allowed spans of I, I_1, I_2 are the same, so for every YES-coloring c of I coloring $c|_{V_1}$ is a YES-coloring of I_1 and $c|_{V_2}$ is a YES-coloring of I_2 . Hence (i) is clear.

For (ii), consider a YES-coloring c_1 of I_1 and a YES-coloring c_2 of I_2 such that $c_1(u) = c_2(w)$ and $c_1(v) = c_2(z)$ and such that for every $x \in V_1 \cap V_2$ we have $c_1(x) = c_2(x)$. Then we define a coloring c

$$c(x) = \begin{cases} c_1(x) & \text{if } x \in V_1 \\ c_2(x) & \text{if } x \in V_2. \end{cases}$$

We know that $c_1(u) = c_2(w)$ and $c_1(v) = c_2(z)$ so all the vertices of $V_1 \cup V_2$ have colors between $c_1(u)$ and $c_1(v)$, i.e., the span of c is at most s as required. It is straightforward to check that c is a proper coloring. \square

Lemma 18 (Lemma 9). *For every (v_L, v_R) -spanned instance $I = (V, d, s)$ of CHANNEL ASSIGNMENT and for every numbers $l, r \in \mathbb{N}$ there exists a (w_L, w_R) -spanned instance $I' = (V \cup \{w_L, w_R\}, d', l + s + r)$ such that*

- (i) *for every YES-coloring c of I there is a YES-coloring c' of I' such that $c'|_V = c$,*
- (ii) *for every YES-coloring c' of I' such that $c'(w_L) \leq c'(w_R)$ we have that*
 - *a coloring $c'|_V$ is a YES-coloring of I ,*
 - *$c'(v_L) = c'(w_L) + l$ and $c'(v_R) = c'(w_R) - r$.*

Proof. We assume that $w_L, w_R \notin V$. We put

$$d'(x, y) = \begin{cases} l + s - 1 + r & \text{for } \{x, y\} = \{w_L, w_R\} \\ l & \text{for } \{x, y\} \cap \{w_L, w_R\} = \{w_L\} \\ r & \text{for } \{x, y\} \cap \{w_L, w_R\} = \{w_R\} \\ d(x, y) & \text{for } x, y \in V. \end{cases}$$

It is straightforward to check that d' satisfies (i) and (ii). \square

Lemma 19 (Lemma 10, proof of **Claim 6**). *Let $G = (V_1 \cup V_2, E, w)$ be a weighted full bipartite graph with nonnegative weights and such that $|V_1| = |V_2|$. Let $n = |V_1|$, $m = \max_{e \in E} w(e)$, $M = n \cdot m + 1$, $l = (4n - 1) \cdot M$ and $s = (8n - 1) \cdot M$. There exists a (v_L, v_R) -spanned instance $I = (V, d, s)$ of CHANNEL ASSIGNMENT with $|V| = O(n)$ and such that for some vertex $v_M \in V$,*

- (i) *for every YES-coloring c of I such that $c(v_L) \leq c(v_R)$ there exists a perfect matching M_G in G such that $c(v_M) = c(v_L) + l + w(M_G)$ and*
- (ii) *for every perfect matching M_G in G there exists a YES-coloring c of I such that $c(v_L) \leq c(v_R)$ and $c(v_M) = c(v_L) + l + w(M_G)$.*

Proof. **Claim 6** Let $\pi : [n] \rightarrow [n]$ be any permutation and $M_\pi = \{v_i^{(1)} v_{\pi(i)}^{(2)} : i \in [n]\}$ be the corresponding perfect matching in G . There is exactly one YES-coloring c such that $\pi_c = \pi$. Moreover $c(v_M) = c(v_L) + l + w(M_\pi)$.

Proof of the claim: Let us consider a sequence of the vertices

$$v_1, a_{\pi(1)}, v_2, w_1, v_3, a_{\pi(2)}, v_4, w_2, v_5, \dots, v_{2n-1}, a_{\pi(n)}, v_{2n}, w_n,$$

$$v_{2n+1}, b_{\pi(1)}, v_{2n+2}, w_{n+1}, v_{2n+3}, b_{\pi(2)}, v_{2n+4}, w_{n+2}, v_{2n+5}, \dots, v_{4n-1}, b_{\pi(n)}, v_{4n}$$

and the coloring c implied by the minimum distances of pairs of consecutive elements in this sequence, i.e., $c(v_1) = 1$, $c(a_{\pi(1)}) = c(v_1) + d(v_1, a_{\pi(1)})$, $c(v_2) = c(a_{\pi(1)}) + d(a_{\pi(1)}, v_2)$, $c(w_1) = c(v_2) + d(v_2, w_1)$, $c(v_3) = c(w_1) + d(w_1, v_3)$, \dots , $c(v_{4n}) = c(b_{\pi(n)}) + d(b_{\pi(n)}, v_{4n})$. We need to check that all the minimum distance constraints d are satisfied and that the span of this coloring is not greater than s .

Note that for every $i \in [4n]$ and for every vertex $x \in V$ such that $v_i \neq x$ we have $d(x, v_i) \geq M$. Therefore for every $i \in [4n - 1]$ we have $c(v_{i+1}) - c(v_i) = (c(v_{i+1}) - c(x)) + (c(x) - c(v_i)) = d(x, v_{i+1}) + d(v_i, x) \geq 2M$ where x is the vertex separating v_i and v_{i+1} in the sequence. Thus for every $i, j \in [4n]$ we have $|c(v_i) - c(v_j)| \geq |i - j| \cdot 2M$ so if $\{i, j\} \neq \{1, 4n\}$ then $|c(v_i) - c(v_j)| \geq d(v_i, v_j)$. Hence also for every $i \in [2n - 1]$ and $j \in [4n]$ we have $|c(w_i) - c(v_j)| = |c(w_i) - c(v_k)| + |c(v_k) - c(v_j)| \geq M + |k - j| \cdot 2M$ where in case that $j \leq 2i$ we have $k = 2i$ and in this case $M + |k - j| \cdot 2M = |4i - 2j + 1| \cdot M$ and in case that $j > 2i$ we have $k = 2i + 1$ and in this case $M + |k - j| \cdot 2M = |2j - 4i - 1| \cdot M = |4i - 2j + 1| \cdot M$ so in both cases $|c(w_i) - c(v_j)| \geq |4i - 2j + 1| \cdot M = d(w_i, v_j)$. We will check the distance between $v_L = v_1$ and $v_R = v_{4n}$ later.

For every $i \in [n]$ and vertex $a_{\pi(i)}$ the closest vertex v_j to the left is v_{2i-1} and to the right is v_{2i} . They are immediate neighbours of $a_{\pi(i)}$ in the sequence so from the definition of c we have $|c(a_{\pi(i)}) - c(v_{2i-1})| = d(v_{2i-1}, a_{\pi(i)})$ and $|c(v_{2i}) - c(a_{\pi(i)})| = d(a_{\pi(i)}, v_{2i})$. Note that for every $j \in [2n]$ we have $d(a_{\pi(i)}, v_j) \leq 2M$ and then for every $j \in [2i - 2]$ we have $|c(a_{\pi(i)}) - c(v_j)| = (c(v_{2i-1}) - c(v_j)) + (c(a_{\pi(i)}) - c(v_{2i-1})) \geq 2M + M > d(a_{\pi(i)}, v_{2i-1})$. Similarly for every $2i + 1 \leq j \leq 2n$ we have $|c(a_{\pi(i)}) - c(v_j)| = ((c(v_{2i}) - c(a_{\pi(i)})) + (c(v_j) - c(v_{2i}))) \geq M + 2M > d(a_{\pi(i)}, v_j)$. For every $2n + 1 \leq j \leq 4n$ we have $|c(v_j) - c(a_{\pi(i)})| = (c(v_{2i}) - c(a_{\pi(i)})) + (c(v_{2n}) - c(v_{2i})) + (c(v_j) - c(v_{2n})) \geq$

$M + 0 + (j - 2n) \cdot 2M = d(a_{\pi(i)}, v_j)$. Because π is a permutation thus we obtain that for every $i \in [n]$ and for every $j \in [4n]$ we have $|c(a_i) - c(v_j)| \geq d(a_i, v_j)$.

For every $i \in [n]$ and $j \in [2n - 1]$ there is at least one vertex v_k with color between the colors $c(a_i)$ and $c(w_j)$ so $|c(a_i) - c(w_j)| = |c(a_i) - c(v_k)| + |c(v_k) - c(w_j)| \geq 2M = d(a_i, w_j)$. For every $i, j \in [n]$ such that $\pi^{-1}(i) < \pi^{-1}(j)$ there are at least two vertices v_k, v_{k+1} with colors $c(a_i) \leq c(v_k) \leq c(v_{k+1}) \leq c(a_j)$. Therefore $|c(a_j) - c(a_i)| = (c(v_k) - c(a_i)) + (c(v_{k+1}) - c(v_k)) + (c(a_j) - c(v_{k+1})) \geq M + 2M + M = 4M = d(a_i, a_j)$.

Similarly we can check that for every $i \in [n]$ and $j \in [4n]$ we have $|c(b_i) - c(v_j)| \geq d(b_i, v_j)$, that for every $i \in [n]$ and $j \in [2n - 1]$ we have $|c(b_i) - c(w_j)| \geq d(b_i, w_j)$ and for every $i, j \in [n]$ such that $i \neq j$ we have $|c(b_i) - c(b_j)| \geq d(b_i, b_j)$.

We need also to check that for every $i \in [n]$ we have $|c(a_i) - c(b_i)| \geq n \cdot 4M = d(a_i, b_i)$. Indeed $|c(b_i) - c(a_i)| = (c(v_{2i}) - c(a_i)) + (c(v_{2n+2i-1}) - c(v_{2i})) + (c(b_i) - c(v_{2n+2i-1})) \geq M + (2n-1) \cdot 2M + M = n \cdot 4M = d(a_i, b_i)$.

Now we are going to deal with the distances between v_L, v_M and v_R . The sum of the minimum color distances of neighbouring elements in the prefix of our sequence:

$$v_1, a_{\pi(1)}, v_2, w_1, v_3, a_{\pi(2)}, v_4, w_2, v_5 \dots, v_{2n-1}, a_{\pi(n)}, v_{2n}, w_n, v_{2n+1}$$

is exactly $2n \cdot 2M + w(M_\pi)$. The sum of the minimum color distances of neighbouring elements in the suffix of our sequence:

$$v_{2n+1}, b_{\pi(1)}, v_{2n+2}, w_{n+1}, v_{2n+3}, b_{\pi(2)}, v_{2n+4}, w_{n+2}, v_{2n+5} \dots v_{4n-1}, b_{\pi(n)}, v_{4n}$$

is exactly $(2n - 1) \cdot 2M + n \cdot m - w(M_\pi)$. So the total sum for the whole sequence is exactly $(4n - 1) \cdot 2M + n \cdot m = s - 1$ and it does not depend on the permutation π . Therefore $|c(v_R) - c(v_L)| = s - 1 = d(v_L, v_R)$. This was the last constraint to check and hence we have shown that c is proper. On the other hand the span of c is s so c is a YES-coloring. Moreover we have $c(v_M) = c(v_L) + (4n - 1) \cdot M + w(M_{\pi_c}) = c(v_L) + l + w(M_{\pi_c})$. Note that all the distances of pairs of consecutive elements of (the whole) sequence are tight, i.e., these distances are equal to the minimum allowed distances for these pairs of the vertices and therefore we cannot decrease any of these distances. On the other hand the span of c is maximum so we cannot increase any of these distances without exceeding the maximum span or violating some of the constraints provided by d . Therefore c is the only one YES-coloring for which the colors of the vertices of this sequence are increasing. Hence c is the only one YES-coloring such that $\pi_c = \pi$. This ends the proof of the claim. \square

Lemma 20 (Lemma 10). *Let $G = (V_1 \cup V_2, E, w)$ be a weighted full bipartite graph with nonnegative weights and such that $|V_1| = |V_2|$. Let $n = |V_1|$, $m = \max_{e \in E} w(e)$, $M = n \cdot m + 1$, $l = (4n - 1) \cdot M$ and $s = (8n - 1) \cdot M$. There exists a (v_L, v_R) -spanned instance $I = (V, d, s)$ of CHANNEL ASSIGNMENT with $|V| = O(n)$ and such that for some vertex $v_M \in V$,*

- (i) *for every YES-coloring c of I such that $c(v_L) \leq c(v_R)$ there exists a perfect matching M_G in G such that $c(v_M) = c(v_L) + l + w(M_G)$ and*
- (ii) *for every perfect matching M_G in G there exists a YES-coloring c of I such that $c(v_L) \leq c(v_R)$ and $c(v_M) = c(v_L) + l + w(M_G)$.*

Proof. Let use Lemma 10 on graph G_1 to obtain a $(v_L^{(1)}, v_R^{(1)})$ -spanned CHANNEL ASSIGNMENT instance $I_1 = (V_1, d_1, s_1)$ with $l_1 = O(n_1^2 m_1)$, $s_1 = 2l_1 + n_1 \cdot m_1 = O(n_1^2 m_1)$ and with the vertex $v_M^{(1)}$ (as in the statement of Lemma 10). The number of the vertices in V_1 is $O(n_1)$.

Similarly, let $I_2 = (V_2, d_2, s_2)$ be a $(v_L^{(2)}, v_R^{(2)})$ -spanned CHANNEL ASSIGNMENT instance with $l_2 = (n_2^2 m_2)$, $s_2 = 2l_2 + n_2 \cdot m_2 = O(n_2^2 m_2)$ and with the vertex $v_M^{(2)}$ obtained from Lemma 10 from graph G_2 . The number of the vertices in V_2 is $O(n_2)$.

Let us identify vertices $v_M^{(1)}$ and $v_M^{(2)}$, i.e., $v_M^{(1)} = v_M^{(2)} = v_M$ and $V_1 \cap V_2 = \{v_M\}$. And let $l_{\max} = \max\{l_1, l_2\} = O(n_1^2 m_1 + n_2^2 m_2)$ and $s = l_{\max} + \max\{s_1 - l_1, s_2 - l_2\} = O(n_1^2 m_1 + n_2^2 m_2)$.

Our span will be s . Note that then every edge with a weight greater than $s - 1$ forces that our instance is a NO-instance. So if we have an edge with a weight greater than s we can replace it with the same edge with but a weight equal to s and the instance will be still a NO-instance. Therefore weights of all our edges will be bounded by $O(n_1^2 m_1 + n_2^2 m_2)$.

We can use Lemma 9 with $l = l_{\max} - l_1$ and with $r = s - (l_{\max} + s_1 - l_1)$ for extending the instance I_1 into a $(w_L^{(1)}, w_R^{(1)})$ -spanned instance $I'_1 (V'_1 = V_1 \cup \{w_L^{(1)}, w_R^{(1)}\}, d'_1, s)$ of CHANNEL ASSIGNMENT.

For every YES-coloring c'_1 of I'_1 we know that $c'_1|_{V_1}$ is a YES-coloring of I_1 and for every YES-coloring c_1 of I_1 there exists a YES-coloring c'_1 of I'_1 such that $c'_1|_{V_1} = c_1$ so from the properties of I_1 (obtained from Lemma 10) we know that

- for every YES-coloring c'_1 of I'_1 such that $c'_1(w_L^{(1)}) \leq c'_1(w_R^{(1)})$ there exists a perfect matching M_1 in G_1 such that $c'_1(v_M) = c'_1(w_L^{(1)}) + l_{\max} + w_1(M_1)$ and
- for every perfect matching M_1 in G_1 there exists a YES-coloring c'_1 of I'_1 such that $c'_1(w_L^{(1)}) \leq c'_1(w_R^{(1)})$ and $c'_1(v_M) = c'_1(w_L^{(1)}) + l_{\max} + w_1(M_1)$.

Similarly we can use Lemma 9 with $l = l_{\max} - l_2$ and with $r = s - (l_{\max} - l_2 + s_2)$ for extending the instance I_2 into a $(w_L^{(2)}, w_R^{(2)})$ -spanned instance $I'_2 (V'_2 = V_2 \cup \{w_L^{(2)}, w_R^{(2)}\}, d'_2, s)$ of CHANNEL ASSIGNMENT such that

- for every YES-coloring c'_2 of I'_2 such that $c'_2(w_L^{(2)}) \leq c'_2(w_R^{(2)})$ there exists a perfect matching M_2 in G_2 such that $c'_2(v_M) = c'_2(w_L^{(2)}) + l_{\max} + w_2(M_2)$ and
- for every perfect matching M_2 in G_2 there exists a YES-coloring c'_2 of I'_2 such that $c'_2(w_L^{(2)}) \leq c'_2(w_R^{(2)})$ and $c'_2(v_M) = c'_2(w_L^{(2)}) + l_{\max} + w_2(M_2)$.

Now we can use Lemma 8 to merge the instances I'_1 and I'_2 into a one $(\{w_L^{(1)}, w_L^{(2)}\}, \{w_R^{(1)}, w_R^{(2)}\})$ -spanned instance $I' = (V'_1 \cup V'_2, d, s)$. A simplified picture of the obtained instance can be found in Figure 4. Note that $V'_1 \cap V'_2 = \{v_M\}$ so

- for every YES-coloring c of I' such that $c(w_L^{(1)}) \leq c(w_R^{(1)})$ there exist perfect matchings M_1 in G_1 and M_2 in G_2 such that $c(v_M) = c(w_L^{(1)}) + l_{\max} + w_1(M_1) = c(w_L^{(1)}) + l_{\max} + w_2(M_2)$, so $w_1(M_1) = w_2(M_2)$ and
- for every two perfect matchings M_1 in G_1 and M_2 in G_2 such that $w_1(M_1) = w_2(M_2)$ there is a YES-coloring c of I such that $c(w_L^{(1)}) \leq c(w_R^{(1)})$ and $c(v_M) = c(w_L^{(1)}) + l_{\max} + w_1(M_1)$.

Therefore the CHANNEL ASSIGNMENT instance I' has a YES-coloring if and only if there are two perfect matchings M_1 in G_1 and M_2 in G_2 such that $w_1(M_1) = w_2(M_2)$. By Lemma 8 and Lemma 9 we know that I' has $O(n_1 + n_2)$ vertices. \square

Corollary 21 (Corollary 13). *There is no algorithm solving CHANNEL ASSIGNMENT in $2^{n \cdot o(\log \log \ell)} \text{poly}(r)$ where n is the number of the vertices and r is the bit size of the instance unless ETH fails.*

Proof. For a given instance of 3-CNF-SAT with n variables and m clauses we use the reduction from Theorem 5 to obtain an instance of COMMON MATCHING WEIGHT with $|V_1| = O\left(\frac{n}{\log n}\right)$, $|V_2| = O\left(\frac{m}{\log m}\right)$ and the maximum matching weights bounded by 2^{3m} .

Then we use the reduction from Lemma 11 to obtain an instance of CHANNEL ASSIGNMENT with

$$n' = O\left(\frac{n}{\log n} + \frac{m}{\log m}\right) = O\left(\frac{n+m}{\log(n+m)}\right)$$

vertices and the weights on the edges bounded by

$$\ell = O\left(\left(\frac{n}{\log n} + \frac{m}{\log m}\right)^2 \cdot 2^{3m}\right).$$

Then $\log \ell = O(n+m)$ and $r = O((n')^2 \cdot \log \ell) = O((n+m)^3)$.

Let us assume that there is an algorithm solving CHANNEL ASSIGNMENT in $2^{n \cdot o(\log \log \ell)} \text{poly}(r)$ -time then we can solve our instance in time

$$2^{O\left(\frac{n+m}{\log(n+m)}\right) \cdot o(\log O(n+m))} \text{poly}(O((n+m)^3)) = 2^{O\left(\frac{n+m}{\log(n+m)}\right) \cdot o(\log(n+m))} \text{poly}(n+m) = 2^{o(n+m)}$$

which contradicts ETH by Corollary 6. \square